Analytic Solutions to the RG Equations of the Neutrino Physical Parameters

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Abstract

In the case of two generation neutrinos, the energy-scale dependence of the lepton-flavor mixing matrix with Majorana phase can be governed by only one parameter r, which is the ratio between the diagonal elements of neutrino mass matrix. By using this parameter r, we derive the analytic solutions to the renormalization group equations of the physical parameters, which are the mixing angle, Majorana phase, and the ratio of the mass-squared difference to the mass squared of the heaviest neutrino. The energy-scale dependence of the Majorana phase is clarified by using these analytic solutions. The instability of the Majorana phase causes in the same parameter region in which the mixing angle is unstable against quantum corrections.

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1 Introduction

Recent neutrino oscillation experiments suggest the strong evidences of the tiny neutrino masses and the lepton flavor mixing [1]-[3]. Studies of the lepton flavor mixing matrix, which we call Maki-Nakagawa-Sakata (MNS) matrix [4], will give us important cues of the physics beyond the standard model. One of the most important studies of the lepton-flavor mixing is the analysis of quantum corrections of the MNS matrix [5]-[11].Especially, the Majorana phase plays the crucial role for stabilities of the mixing angle against quantum corrections [8].

In this paper we analyze the energy-scale dependence of the MNS matrix with the physical Majorana phase in two generation neutrinos. According to the LSND experiment[3], the scenario of two heavy degenerate neutrinos can be realistic, where we can neglect the first generation effects in the energy-scale dependence of the MNS matrix. In this scenario, the MNS matrix has two parameters, which are the mixing angle and the Majorana phase. The energy-scale dependence of them are completely determined by only one parameter r which is the ratio between the diagonal elements of neutrino mass matrix. The energy-scale dependence of r can be easily solved. When we use this parameter r, we can easily obtain the analytic solution to the energy-scale dependence of physical parameters in the MNS matrix which are the mixing angle and the Majorana phase. The energy-scale dependence of $\Delta m_{23}^2/m_3^2$ is also determined by only one parameter r, where Δm_{23}^2 is the mass-squared difference between the second and the third generations and m_3 is the absolute value of the heaviest neutrino mass. The energy-scale dependence of the Majorana phase is clarified by using these analytic solutions. The instability of the Majorana phase causes in the same parameter region in which the mixing angle is unstable against quantum corrections.

2 Energy-Scale Dependence of κ

The effective Yukawa couplings in the lepton sector are given by

$$\mathcal{L}_{yukawa}^{\text{low}} = y^{\text{e}}{}_{ij}\phi_{\text{d}}L_i \cdot e_{Rj}^c - \frac{1}{2}\kappa_{ij}(\phi_{\text{u}}L_i)\cdot(\phi_{\text{u}}L_j) + h.c. \,, \tag{1}$$

where ϕ_{u} and ϕ_{d} are the SU(2)$_L$ doublet Higgs bosons that give Dirac masses to the up-type and down-type fermions, respectively. L_i is the i-th generation SU(2)$_L$ doublet lepton. e_{Ri} is the i-th generation right-handed charged-lepton. The matrix κ induces the neutrino Majorana mass matrix. y^{e} is the Yukawa matrix of the charged-lepton. In this paper, we take the diagonal base of y^{e}. Once y^{e} is taken diagonal at a certain scale, the diagonality of it is kept at all energies in the one-loop level. This is because there are no lepton-flavor-mixing terms, except for κ in the minimal supersymmetric standard model (MSSM). In this base the matrix κ is diagonalized as

$$\kappa_D = U^T \kappa\, U \,, \tag{2}$$

where U is an unitary matrix (the MNS matrix) and κ_D is the diagonalized mass matrix. The MNS matrix U is parameterized as,

$$U = e^{ix}\begin{pmatrix} 1 & 0 \\ 0 & e^{iy} \end{pmatrix}\begin{pmatrix} \cos\theta_{23} & \sin\theta_{23} \\ -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}\,, \tag{3}$$

where θ_{23} is the mixing angle and ϕ is the Majorana phase. The phases of x and y are unphysical parameters, which can be rotated out by the field redefinition.

Since κ is complex and symmetric matrix, κ can be parameterized as [7]

$$\kappa = \lambda \begin{pmatrix} r & c\sqrt{r}e^{i\chi} \\ c\sqrt{r}e^{i\chi} & 1 \end{pmatrix}. \tag{4}$$

In this parameterization, c and χ are non-negative values and energy-scale independent parameters. The energy-scale dependence of the κ can be controlled by two parameters r and λ. Since the λ is overall factor, the energy-scale dependence of the mixing angle and the Majorana phase in the MNS matrix are governed by only one parameter r.

The renormalization group equation for r is given as [7, 9]

$$\frac{d}{dt} \ln r = -\frac{1}{8\pi^2}(y_\tau^2 - y_\mu^2), \tag{5}$$

where t is the scaling parameter which is related to the renormalization scale μ as $t = \ln \mu$. y_τ and y_μ are Yukawa couplings of the charged-leptons, τ and μ, respectively. We can obtain the solution to eq.(5) as

$$r(m_R) = r(m_Z) (1 - \epsilon)^2 , \tag{6}$$

where m_R is a certain high energy scale. When $\tan\beta \leq 50$, where $\tan\beta$ is the ratio of two vacuum expectation values of Higgs bosons $\tan\beta = \langle\phi_u\rangle/\langle\phi_d\rangle$, the parameter ϵ can be obtained approximately [9] as

$$\epsilon \simeq 1 - \exp\left(-\frac{1}{16\pi^2} \int_{\ln m_Z}^{\ln m_R} \left(y_\tau^2 - y_\mu^2\right) dt\right) ,$$

$$\simeq \frac{y_\tau^2}{16\pi^2} \ln\left(\frac{m_R}{m_Z}\right). \tag{7}$$

Figure 1 shows the $\tan\beta$ dependence of ϵ at $m_R = 10^6$GeV and $m_R = 10^{13}$GeV. Since the energy-scale dependence of r is very small, the difference between the values of $r(m_R)$ and $r(m_Z)$ is not so large [7, 9]. For instance, in the case of $\tan\beta = 20$ and $m_R = 10^{13}$ GeV, ϵ is nearly equal to 0.005.

3   Analytic Solutions to the RG Equations

By using the energy-scale dependent parameters r and λ, and the energy-scale independent parameters c and χ, the masses of neutrinos, ν_2 and ν_3, are given as

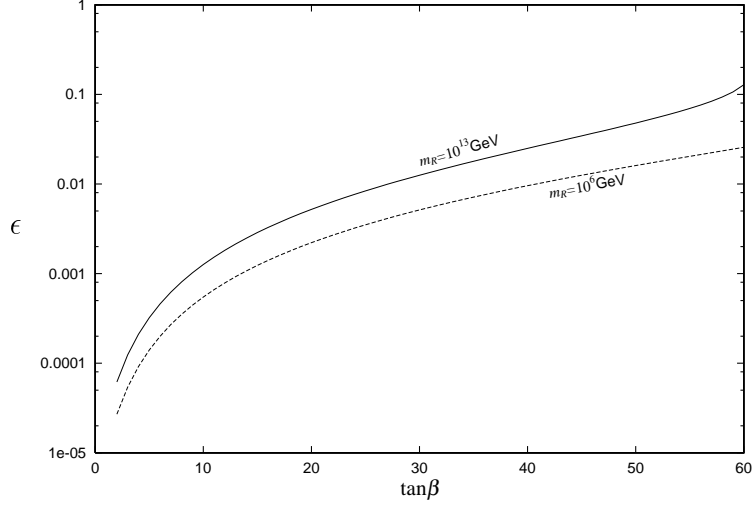$$m_2 = \frac{\lambda v^2 \sin^2\beta}{4}\sqrt{2\left\{r^2 + 1 + 2c^2r - f(r, c, \chi)\right\}},$$

Figure 1: $\tan\beta$ dependence of the ϵ at $m_R = 10^6$GeV and $m_R = 10^{13}$GeV.

$$(8)$$

$$m_3 = \frac{\lambda v^2 \sin^2\beta}{4}\sqrt{2\left\{r^2 + 1 + 2c^2 r + f(r,c,\chi)\right\}}.$$

The function $f(r,c,\chi)$ is defined as

$$f(r,c,\chi) \equiv \sqrt{(1-r^2)^2 + 4c^2 r\left\{g(r,\chi)\right\}^2} , \tag{9}$$

where the function $g(r,\chi)$ is defined as

$$g(r,\chi) \equiv \left| re^{-i\chi} + e^{i\chi} \right|$$

$$= \sqrt{(r-1)^2 + 2r(\cos 2\chi + 1)}. \tag{10}$$

The mass-squared difference between neutrinos is given as

$$\Delta m_{23}^2 = m_3^2 - m_2^2 = \lambda^2 f(r,c,\chi)\frac{v^4 \sin^4\beta}{4} . \tag{11}$$

The mass-squared difference scaled by the heaviest neutrino mass m_3 can be given as

$$\frac{\Delta m_{23}^2}{m_3^2} = \frac{2f(r,c,\chi)}{(r-1)^2 + 2r(c^2+1) + f(r,c,\chi)} . \tag{12}$$

By using the parameters, c, χ, and r, the mixing angle θ_{23} and the Majorana phase ϕ can be written as

$$\sin^2 2\theta_{23} = \frac{4c^2 r\{g(r,\chi)\}^2}{\{f(r,c,\chi)\}^2} ,$$

$$= \cfrac{1}{1 + \cfrac{(r-1)^2(r+1)^2}{4c^2 r\{(r-1)^2 + 2r(1 + \cos 2\chi)\}}} \ , \tag{13}$$

and

$$\cos\phi = \frac{\cos 2\chi - c^2 + \cfrac{2r}{\{g(r,\chi)\}^2} \sin^2 2\chi}{\sqrt{c^4 - 2c^2 \cos 2\chi + 1}} \ . \tag{14}$$

4 Energy-Scale Dependence of the Physical Parameters

The energy-scale dependent parameter is only r in the physical parameters of eqs.(12), (13) and (14). Thus studying the energy-scale dependence of the physical parameters is equivalent to studying the r dependence of them.

At the low energy-scale, the large mixing between the second and the third generations, $\sin^2 2\theta_{23} \simeq 1$ is favored by the atmospheric neutrino experiments [2]. This means that

$$\frac{(r-1)^2(r+1)^2}{4c^2 r\{(r-1)^2 + 2r(1 + \cos 2\chi)\}} \ll 1 \tag{15}$$

should be required at the low energy-scale from the eq.(13). There are following two cases which satisfy the eq.(15):

(a) The parameter c is very large such as

$$c^2 \gg \frac{(r-1)^2(r+1)^2}{4r\{(r-1)^2 + 2r(1 + \cos 2\chi)\}} \ . \tag{16}$$

In this case, the energy-scale dependence of the physical parameters is negligible, because the dominant component of the κ is the energy-scale independent parameter c. However, even if the parameter c is enough large but finite, eq.(16) is not satisfied in the vicinity of $r = 0$. In this parameter region, the large mixing is unstable against quantum corrections. When r is in the vicinity of the zero, the parameter c must be infinitely large in order to satisfy eq.(16). Equation(16) can be rewritten as

$$c\sqrt{r} \gg \frac{(r-1)(r+1)}{2g(r,\chi)} \ , \tag{17}$$

where $c\sqrt{r}$ is the absolute value of the off-diagonal elements in the matrix κ. When eq.(17) is satisfied, the large mixing is always preserved against quantum corrections [9, 11]. This case corresponds with so-called pseudo-Dirac type neutrino mass matrix [12].
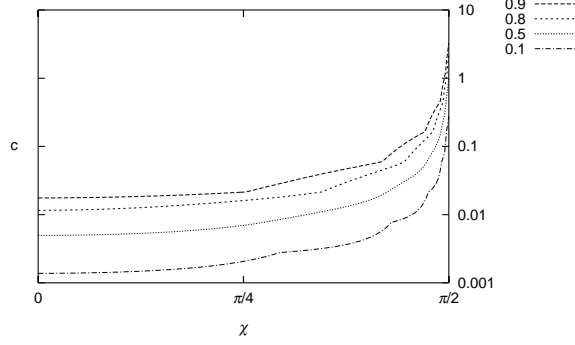
Figure 2: The contour plot of $\sin^2 \theta_{23}$ at $\epsilon = 0.005$ ($m_R = 10^{13}$GeV and $\tan \beta = 20$).

(b) The parameter r is nearly equal to 1 with $\chi \neq \pi/2$. We discuss the case of $\chi = \pi/2$ later. In this case, it is necessary that we study the r dependence of the physical parameters in the vicinity of the $r = 1$, because mixing angle $\sin^2 2\theta_{23}$ is not stable against quantum corrections in this region [10]. Since the large c makes the mixing angle to be stable against quantum corrections as we have seen in the case (a), we study the region of $c < 10$. Here we input $\sin^2 2\theta_{23} = 1$ at m_Z scale. Figure 2 shows the contour plot of $\sin^2 2\theta_{23}$ at m_R scale ($\epsilon = 0.005$) for the continuous changes of c and χ. The more the value deviates from 1 the larger the change of the mixing angle by quantum corrections becomes. As the values of χ approach $\pi/2$ and the parameter c becomes small, the $\sin^2 2\theta_{23}$ becomes unstable against quantum correction. When we define $\alpha \equiv \pi/2 - \chi$ and take the parameters, c, ϵ and α to be small, the left-hand side of eq.(15) can be rewritten as

$$\frac{(r-1)^2(r+1)^2}{4c^2r\{(r-1)^2 + 2r(1+\cos 2\chi)\}} \sim \frac{\epsilon^2}{c^2\alpha^2} \ . \tag{18}$$

This indicates that if the $c\alpha$ is much smaller than ϵ, eq.(15) is not satisfied. Therefor the mixing angle is unstable against the quantum corrections in this region. This result is the same as that of Ref. [9, 10, 11].

In order to understand the physical meaning of the parameters $r(\epsilon)$, c and $\chi(\alpha)$ obviously, these parameters should be related to the other physical parameters, the Majorana phase and the mass-squared difference. Especially, it is necessary to investigate the physical meaning of the parameter regions in which the mixing angle $\sin^2 2\theta_{23}$ is unstable against quantum corrections.

Figure 3 shows the Majorana phase ϕ in eq.(14) in the vicinity of $\chi = \pi/2$ for $c = 1$ and $c = 0.1$ cases, when we take $\epsilon = 0$ and $\epsilon = 0.005$ for both cases. We set the $\sin^2 2\theta_{23} = 1$ at the m_Z scale. Figure 3 shows that the Majorana phase is not changed by the quantum corrections, when the χ is not close to $\pi/2$. By using small parameter α, the Majorana phase is obtained as

$$\cos \phi = \frac{1 - c^2}{1 + c^2} \left(1 + \frac{2c^2}{(c^2+1)^2}\alpha^2 \right) \tag{19}$$

for $r = 1$. By using this equation, we obtain the Majorana phase at the m_Z scale for the large mixing. On the other hand, when the χ is close to the $\pi/2$, the Majorana phase is changed by the quantum corrections and it always becomes close to -1. The mixing angle
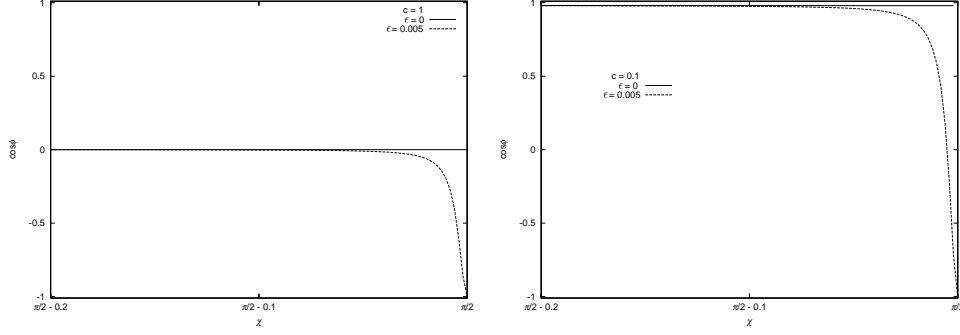
Figure 3: The plot of the Majorana phase ϕ at $\epsilon = 0$ and $\epsilon = 0.005$.

is unstable against quantum corrections in the region of small c and $\chi \simeq \pi/2$. Comparing to Figure 2 and Figure 3, we notice that the instability of the Majorana phase causes in the same parameter region in which the mixing angle is unstable against quantum corrections.

Figure 4 shows the contour plot of $\Delta m_{23}^2/m_3^2$ in eq.(12) at $\epsilon = 0$ and $\epsilon = 0.005$ for the continuous changes of c and χ. According to the Figure 4, the parameter region in which the mixing angle is unstable against quantum corrections is corresponding to the region in which the masses are more degenerate. This result is consistent with that of Ref. [10].
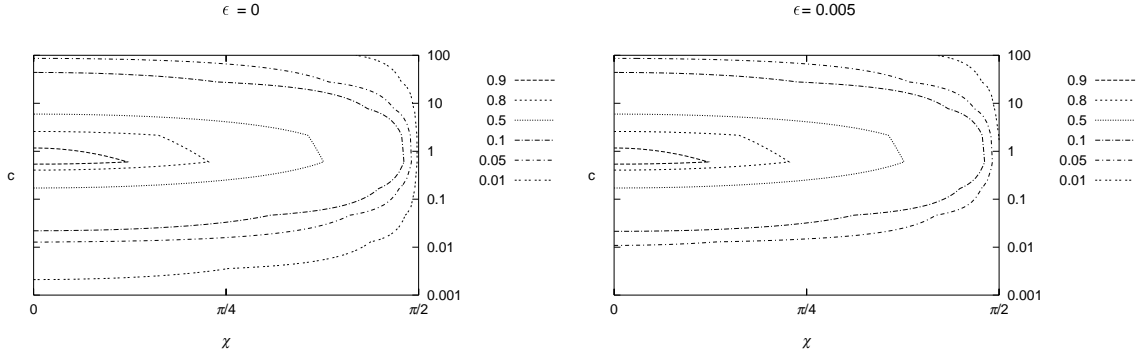

Figure 4: The contour plots of $\Delta m_{23}^2/m_3^2$ at $\epsilon = 1$ and $\epsilon = 0.005$.

Finally, we discuss the special case of $\chi = \pi/2$, where the r dependence of the mixing angle and the Majorana phase ϕ are quite different from the case of $\chi \neq \pi/2$. In this case eq.(13) becomes

$$\sin^2 2\theta_{23} = \frac{1}{1 + \frac{(r+1)^2}{4c^2 r}} . \tag{20}$$

This equation indicates that $\sin^2 2\theta_{23}$ does not automatically take the maximal mixing at $r = 1$. It is because the mixing angle becomes

$$\sin^2 2\theta_{23} = \frac{c^2}{1 + c^2} \tag{21}$$

when $r = 1$. Thus the large mixing angle is only realized when $c \gg 1$ (case (a)) where the large mixing is stable against quantum corrections. For the Majorana phase, we can obtain $\cos\phi = -1$ from eq.(14) at $\chi = \pi/2$. In this case, the Majorana phase does not depend on r and c. Therefore the Majorana phase is independent of the energy-scale.

5 Summary

In this paper we study the energy-scale dependence of the MNS matrix with the physical Majorana phase in two generation neutrinos. According to the LSND experiment[3], the scenario of two heavy degenerate neutrinos can be realistic and important, where the first generation effects are neglected in the energy-scale dependence of the MNS matrix. In this case, the MNS matrix has two physical parameters, which are the mixing angle θ_{23} and the Majorana phase ϕ. The energy-scale dependence of these parameters are controlled by only one parameter r which is the ratio of the diagonal elements in the neutrino mass matrix [7]. This r also governs the mass-squared difference scaled by the heaviest neutrino mass. We can easily solve the renormalization group equation of this parameter, r. Then the analytic solutions to the energy-scale dependence of the physical parameters can be obtained. Especially, the energy-scale dependence of the Majorana phase is clarified by using these analytic solutions. The instability of the Majorana phase causes in the same parameter region in which the mixing angle is unstable against quantum corrections.

Acknowledgment

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